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Schwartz Space and Differentiability of Fourier Transform

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Abstract: **One of the reasons that the Fourier transform is so useful is that it converts operations on function f in to different operations on the function $\hat{f}$.**

In this paper we discuss some properties of differentiability of Fourier transform and we generalize the results that are stated by many other authors, and we give some propriety in Schwartz's space S using the main result.
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1. PRELIMINARIES AND INTRODUCTION

In this section we begin by some properties of Fourier transform. Let p be any real number, $p \geq 1$, and $L_p(-\infty,+\infty)$ the vector space of all complex valued functions $f(x)$ of all real variable x , $-\infty < x < +\infty$, such that f is lebesgue measurable, and

$$\|f\|_p = \left(\int_{-\infty}^{+\infty} |f(x)|^p \, dx\right)^{\frac{1}{p}} < +\infty. \quad (1.1)$$

We call the number $\|f\|_p$ the L_p -norm of f .

Now if $f(x) \in L_1(-\infty < x < +\infty)$ and α any real number, we define the Fourier transform of f by:

$$\hat{f}(\alpha) = \int_{-\infty}^{+\infty} f(x) e^{i\alpha x} dx. \quad (1.2)$$

And say that $\hat{f}$ is the Fourier transform of $f \in L_1(-\infty,+\infty)$.We write symbolical

$$\hat{f} = F[f] \quad or \quad \hat{f}(v) = F[f(x)].$$

In the special case when f is even, $f(-x) = f(x)$ for all real values of x , (1. 2) take the form

$$\hat{f}(\alpha) = 2\int_{0}^{+\infty} f(x) \cos \alpha x \, dx. \quad (1.3)$$

If f is odd, $f(-x) = -f(x)$ for all real values of x , (1. 3) take the form

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$$-i\,\hat{f}(\alpha) = 2\int_{0}^{+\infty} f(x)\sin\alpha x\,dx \,. \quad (1.4)$$

Now we give some basic properties of Fourier transform of functions in $L_1(-\infty,+\infty)$ (see [1-8]).

A) If $f(x)\in L_1(-\infty<x<+\infty)$, than $\hat{f}$ is bounded on $(-\infty,+\infty)$, since for all real α, we have

$$\left|\hat{f}(\alpha)\right| \le \int_{-\infty}^{+\infty}|f(x)|\,dx = \|f\|_1 < +\infty. \quad (1.5)$$

where $\|f\|_1$ denotes the L_1-norm of f, so that

$$\sup_{-\infty<x<+\infty}\left|\hat{f}(\alpha)\right| \le \|f\|_1 < +\infty. \quad (1.6)$$

B) If $f(x)\in L_1(-\infty<x<+\infty)$, than $\hat{f}(\alpha)$ is continuous on $(-\infty,+\infty)$, for if h is a real number, $h\neq 0$, then

$$\left|\hat{f}(\alpha+h)-\hat{f}(\alpha)\right| = \left|\int_{-\infty}^{+\infty}\left[f(x)e^{i(\alpha+h)x} - f(x)e^{i\alpha x}\right]dx\right|$$

$$= \left|\int_{-\infty}^{+\infty} f(x)\,e^{i\alpha x}(e^{ihx}-1)\,dx\right|$$

$$\le \int_{-\infty}^{+\infty}|f(x)|\left|e^{ihx-1}\right|dx \to o \text{ as } h\to o. \quad (1.7)$$

and hence $\hat{f}$ is continuous at point α, Where $-\infty<\alpha<+\infty$.

c)The operator $f\to\hat{f}$ is linear in the sense that

$$(c_1f_1+c_2f_2)^{\wedge}(\alpha) = c_1\hat{f}_1(\alpha) + c_2\hat{f}_2(\alpha)$$

where c_1, c_2 are complex numbers and $f_1, f_2\in L_1(-\infty,+\infty)$.

D) Let h be a fixed real number, and $f(x)\in L_1(-\infty<x<+\infty)$. Then the Fourier transform of $f(x+h)$, is the translation of

$f(x)$ by h, equals $\hat{f}(\alpha)e^{-i\alpha h}$.

E) Let t be a fixed real number and $f(x)\in L_1(-\infty<x<+\infty)$. Then the Fourier transform of $f(x)e^{itx}$ is $\hat{f}(\alpha+t)$.

F) Let λ be a fixed real number, $\lambda\neq 0$, and $f(x)\in L_1(-\infty<x<+\infty)$. Then the Fourier transform of $f(\lambda x)$ is

$$\frac{1}{|\lambda|}\,\hat{f}\left(\frac{\alpha}{\lambda}\right).$$

G) If $\bar{f}$ denotes the complex conjugate of f and $f\in L_1(-\infty,+\infty)$, then the Fourier transform of $\bar{f}(x)$ is $\overline{\hat{f}(-\alpha)}$ since the complex conjugate of

$$\int_{-\infty}^{+\infty} \bar{f}(x) e^{i\alpha x} dx \text{ equals } \hat{f}(-\alpha).$$

H) If $f \in L_1(-\infty, +\infty)$ and $f_n \in L_1(-\infty, +\infty)$ for n = 1,2,………, and

$\| f_n - f \|_1 \to 0,\ as\ n \to +\infty,$ then we have

$$\lim_{x\to\infty} \hat{f}_n(\alpha) = \hat{f}(\alpha). \qquad (1.8)$$

uniformly for $-\infty < \alpha < +\infty$. In effect, by (1.5) and (1.8), we have

$$\sup_{-\infty<\alpha<+\infty} \left| \hat{f}_n(\alpha) - \hat{f}(\alpha) \right| \leq \| f_n - f \|_1 .$$

J) If $f_1, f_2 \in L_1(-\infty, +\infty)$, then

$$\int_{-\infty}^{+\infty} \hat{f}_1(y)\, f_2(y) dy = \int_{-\infty}^{+\infty} f_1(y)\, \hat{f}_2(y) dy. \quad (1.9)$$

which is called composition rule.

K) B. Muckenhoupt posed in [10], the problem of characterizing those non-negatives functions u and v ,which for some p , $1 \leq p < \infty$,the inequality

$$\int_{-\infty}^{\infty} \left| \hat{f}(x) \right|^p u(x) dx \leq c \int_{-\infty}^{\infty} |f(x)|^p v(x) dx \ ,$$

Holds for any f , and in [11] deal only with the case where either $u \equiv 1$ or $v \equiv 1$, finding that when $v \equiv 1$, $1 < p < 2$, a

Necessary condition is that for any $r > 0$

$$\left[\sum_{k=-\infty}^{k=+\infty} \left(\int_{rk}^{r(k+1)} u(x) dx \right)^b \right]^{\frac{1}{b}} \leq c r^{p-1}$$

where $b = \dfrac{2}{2-p}$, and that a sufficient condition $(v \equiv 1\, , 1 \leq p)$ is that for any measurable set E , we have :

$$\int_E u(x) dx \leq c |E|^{p-1}.$$

Many authors studied this subject, for instance the reader should refer to [1-9].

2. THE MAIN RESULTS

In this section, we prove the main theorems

Theorem 2.1:

Let $f(x) \in L_1(-\infty < x < +\infty)$ and $x^m f(x) \in L_1(-\infty < x < +\infty)$, 20where m is a positive integer. Then $\hat{f}(\alpha)$ is a continuously differentiable m times for $-\infty < x < +\infty$, and we have

$$\left(\hat{f} \right)^{(m)}(\alpha) = \int_{-\infty}^{+\infty} \left[(ix)^{(m)} f(x) \right] e^{i\alpha x} dx \ .$$

So that

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$$\left|\left(\hat{f}\right)^{(m)}(\alpha)\right| \leq \int_{-\infty}^{+\infty} \left|(ix)^{(m)} f(x)\right| dx .$$

Proof:

We are going to use induction on m , we assume that the result is true for all n such that $1 \leq n \leq m$.

Consider the function

$$\hat{F}(\alpha) = \left(\hat{f}\right)^{(n)}(\alpha) = \int_{-\infty}^{+\infty} F(x)\, e^{i\alpha x} dx. \quad (2.1)$$

And also

$$F_h(x) = F(x)\left(\frac{e^{ihx}-1}{h}\right)$$

for h real and $h \neq 0$. Then

$$F_h(\alpha) = \frac{\hat{F}(\alpha+h)-\hat{F}(\alpha)}{h}. \quad (2.2)$$

Now

$$\begin{aligned}\lim_{h\to 0} F_h(x) &= \lim_{h\to 0} F(x)\left(\frac{e^{ihx}-1}{h}\right)\\ &= F(x)\lim_{h\to 0}\left(\frac{e^{ihx}-1}{h}\right)\\ &= ixF(x). \qquad (2.3)\end{aligned}$$

So $F_h(x) \to ix\,F(x)$ point wisely, for almost every x , as $h \to 0$ and

$$\begin{aligned}\left|F_h(x)\right| &= \left|F(x)\left(\frac{e^{ihx}-1}{h}\right)\right|\\ &\leq \left|F(x)\right|\left|\frac{e^{ihx}-1}{h}\right|. \qquad (2.4)\end{aligned}$$

Note that

$$\left|\frac{e^{ihx}-1}{h}\right| \leq |x|. \qquad (2.5)$$

Let $M(t) = e^{ith} - 1$. By applying Mean-Value theorem on $M(t)$, $\forall t \in [0,1]$, we have

$$M'(t) = ihe^{ith} = \frac{e^{iht}-1}{x}.$$

Then

$$\left|\frac{e^{iht}-1}{x}\right| = \left|ixe^{ixt}\right| \leq |x|.$$

We drive from (2.4) and by hypothesis that

$$\left|F_h(x)\right| \leq |x|\left|F(x)\right| \in L_1(-\infty,+\infty)$$

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Hence

$$\left|F_h(x) - ixF(x)\right| \le \left|F_h(x)\right| + \left|ixF(x)\right|$$
$$\le 2|x||F(x)| \in L_1(-\infty,+\infty)$$

By Riemann-Lebesgu's Theorem on dominated convergence, we have

$F_h(x) \to ixF(x)$ in the L_1- norm, that is

$$\left\|F_h - ixF\right\|_1 \to 0 \;\; as \;\; h \to 0.$$

By property (g) we get

$$\hat{F}_h(x) \to \int_{-\infty}^{+\infty} (ixF(x))e^{i\alpha x}dx\,.$$

Uniformly convergent.

The last integral is a bounded continuous function of α , $-\infty < \alpha < +\infty$.

Now

$$\lim_{h\to 0} F_h(\alpha) = \lim_{h\to 0} \frac{\hat{F}(\alpha+h) - \hat{F}(\alpha)}{h}$$

$$= \left((\hat{F})(\alpha)\right)'$$

$$= \left(\left(\hat{f}\right)^{(n)}\right)'(\alpha)\;\;.$$

$$= \left(\hat{f}\right)^{(n+1)}(\alpha).$$

$$(\hat{F})'(\alpha) = \left(\hat{f}\right)^{(n+1)}(\alpha)$$

Hence

$$= \int_{-\infty}^{+\infty} \left((ix)^{(n+1)} f(x)\right)e^{i\alpha x}dx,\, 1 \le n \le m-1.$$

this shows that the result is true for any positive integer m .

Therefore

$$\left(\hat{f}\right)^{(m)}(\alpha) = \int_{-\infty}^{+\infty} \left((ix)^{(m)} f(x)\right)e^{i\alpha x}dx\;.$$

However

$$\left|\left(\hat{f}\right)^{(m)}(\alpha)\right| = \left|\int_{-\infty}^{+\infty} \left((ix)^{(m)} f(x)\right)e^{i\alpha x}dx\right|$$

$$\le \int_{-\infty}^{+\infty} \left|(ix)^{(m)} f(x)\right|\left|e^{i\alpha x}\right|dx\,.$$

$$\le \int_{-\infty}^{+\infty} \left|(ix)^{(m)} f(x)\right|dx\,.$$

Hence the proof is complete. ■

Theorem 2.2:

Let $f(x) \in L_1(-\infty < x < +\infty)$, f is continuously differentiable m times and

$$f^{(r)}(x) \in L_1(-\infty < x < +\infty), \quad for \quad 0 \le r \le m.$$

Then

$$\left(f^{(m)}\right)^{\wedge}(\alpha) \;=\; (-i\,\alpha)^m \hat{f}(\alpha)$$

so that

$$\left|\alpha^m \hat{f}(\alpha)\right| \;\le\; \int_{-\infty}^{+\infty} \left|f^{(m)}(x)\right| dx.$$

Proof :

First, we calculate the formula $\int_{-\infty}^{-\infty} f(x) e^{i\alpha x} dx$.

Let α be real number, For $R > 0$, $\alpha \ne 0$, we have

$$\int_{-R}^{-R} f(x) e^{i\alpha x} dx = \left[f(x) \frac{e^{i\alpha x}}{i\alpha} \right]_{-R}^{+R} - \int_{-R}^{+R} \frac{e^{i\alpha x}}{i\alpha} f'(x)\, dx$$

Now $f(x)$ tends to finite limits as $x \to \pm\infty$, then

$$f(x) - f(0) = \int_0^x f'(x)\, dx \quad and \quad f' \in L_1(-\infty, +\infty).$$

So that

$$f(\pm\infty) \;=\; f(0) + \int_{-R}^{R} f'(t)\, dt. \qquad (2.6)$$

But $f(\pm\infty) = 0$, hence letting $R \to +\infty$ in (2.1), we have

$$\int_{-\infty}^{-\infty} f(x) e^{i\alpha x} dx = -\int_{-\infty}^{+\infty} \frac{e^{i\alpha x}}{i\alpha} f'(x)\, dx \;\; \alpha \ne 0. \qquad (2.7)$$

By the same argument, we have

$$\int_{-\infty}^{-\infty} f(x) e^{i\alpha x} dx = (-1)^m \int_{-\infty}^{+\infty} \frac{e^{i\alpha x}}{(i\alpha)^m} f^{(m)}(x)\, dx \;\; \alpha \ne 0. \qquad (2.8)$$

this implies that

$$(-i\,\alpha)^m \hat{f}(\alpha) \;=\; \left(f^{(m)}\right)^{\wedge}(\alpha) \quad for \quad \alpha \ne 0.$$

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This holds also for $\alpha = 0$ by continuity (since the left hand side is zero and the right side is $\int_{-\infty}^{+\infty} f^{(m)}(x)dx = 0$).

Now

$$\left|(-ix)^m \hat{f}(\alpha)\right| = \left|\alpha^m \hat{f}(\alpha)\right| \le \int_{-\infty}^{+\infty} \left|f^{(m)}(x)\right| d$$

$$\le \left\| f^{(m)} \right\|_1 < +\infty. \qquad (2.9)$$

By Riemann-Lebesgue lemma (which says that the Fourier transform of f tends to 0 as z tends to infinity) and by (2.9) also f is an arbitrary measurable function, and since $f^{(m)} \in L_1(-\infty,\infty)$, we conclude that

$$\hat{f}(\alpha) = 0\left(\frac{1}{|\alpha|^m}\right) \quad as \quad |\alpha| \to +\infty .$$

Hence the proof is complete. ■

Theorem 2.3 :

If $\hat{f} \in L_1(-\infty,\infty)$, then the integral $\frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{f}(\alpha)e^{-i\alpha x}d\alpha$ defines a continuous function of x .

Proof :

Let

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{f}(\alpha)e^{-i\alpha x}d\alpha \quad ,$$

then consider

$$f(x+h)-f(\alpha)=\frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{f}(\alpha)e^{-i\alpha x}\left[e^{-i\alpha h}-1\right]d\alpha$$

$$\left|f(x+h)-f(\alpha)\right| =$$

$$=\left|\frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{f}(\alpha)e^{-i\alpha x}\left[e^{-i\alpha h}-1\right]d\alpha\right|$$

$$\le \frac{1}{2\pi}\int_{-\infty}^{+\infty} \left|\hat{f}(\alpha)\left[e^{-i\alpha h}-1\right]\right|d\alpha.$$

But

$$\left|\hat{f}(\alpha)\right|\left|e^{-i\alpha h}-1\right| \le 2\left|\hat{f}(\alpha)\right| \in L_1(-\infty,+\infty),$$

and

$$\left|\hat{f}(\alpha)\right|\left|e^{-i\alpha h}-1\right| \to 0 \;\; as \;\; h \to 0.$$

For almost all $\alpha \in L_1(-\infty,+\infty)$. It follows from Riemann-Lebesgue,s Theorem on dominated convergence that

$$\int_{-\infty}^{+\infty} \left|\hat{f}(\alpha)\right|\left|e^{-i\alpha h}-1\right|d\alpha \to 0 \quad as \quad h \to 0,$$

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and hence f is continuous function at the point x, where $x \in (-\infty, +\infty)$. Hence the proof is complete. ■

Remark 2.4:

We indicate that if $f(x) \in L_1(-\infty < x < +\infty)$, it dose note necessary that the Fourier transform $\hat{f}(\alpha)$ belongs to $L_1(-\infty < \alpha < +\infty)$, we can see in the following example:

Taking

$$f(x) = \begin{cases} 1, \; for \; |x| \le 1 \\ \\ 0, \; for \; |x| > 1 \end{cases}$$

It is clearly that $f(x) \in L_1(-\infty < x < +\infty)$, but for α any real, $-\infty < \alpha < +\infty$, we have

$$\hat{f}(\alpha) = \int_{-\infty}^{+\infty} f(x) e^{i\alpha x} dx$$

$$= \int_{-1}^{+1} e^{i\alpha x} dx$$

$$= 2\int_{o}^{+1} \cos \alpha x \; dx$$

$$= 2 \left. \frac{\sin \alpha x}{\alpha} \right|_{x=0}^{x=1}$$

$$= 2 \frac{\sin \alpha}{\alpha}.$$

Note that here $\hat{f}(\alpha) \notin L_1(-\infty < \alpha < +\infty)$.

Definition 2.1 :

We say that $f : \mathrm{X} \to Y$ has a compact support, if f vanishes outside some compact subset of X, where X, Y are topological spaces.

Now apply [12] and theorem 2.1, we have the following theorem.

Theorem 2.4 :

Let f has a compact support in C, and define

$$\forall x \in C : F(\alpha) = \int_{-\infty}^{+\infty} f(\alpha) e^{i\alpha x} dx \,,$$

then $F(\alpha)$ is analytic in C.

Proof:

Firstly we prove that $F(\alpha)$ defines a continuous function. Suppose that

$\operatorname{supp} f \subset [-R, R]$. Then $\forall x \in \operatorname{supp} f$, we have $\left| e^{i\alpha x} \right| \le e^{R|\zeta \alpha|}$, so that if $h \to 0$, then

$$\left| F(\alpha + h) - F(\alpha) \right| = \left| \int_{-\infty}^{+\infty} e^{i\alpha x} (e^{i\alpha h} - 1) f(x) dx \right|$$

and note that, for small $|h|$,

$\left|e^{i\alpha x}(e^{i\alpha h}-1)f(x)\right| \le (R|h|)e^{R(|\xi\alpha|+1)}\,|f(x)| \to 0$ so

$\left|e^{i\alpha x}(e^{i\alpha h}-1)f(x)\right| \le (R|h|)e^{R(|\xi\alpha|+1)}|f(x)|$

which is integrable and so the dominated convergence theorem tells us that F is continuous.

We now claim that F is analytic. For this let

$$g(\alpha) = \int_{-\infty}^{\infty}(ix)e^{i\alpha x}f(x)dx \ .$$

Note that g makes sense because f has compact support .To show that F is analytic

we only need to show that $\lim\limits_{|h|\to 0}\dfrac{|F(\alpha+h)-F(\alpha)-g(\alpha)|}{|h|}=0,\ \forall\alpha\in\mathbb{C}$, let $\alpha\in\mathbb{C}$, and

consider $\dfrac{|F(\alpha+h)-F(\alpha)-g(\alpha)|}{|h|} = \dfrac{1}{|h|}\left|\int_{-\infty}^{\infty}(e^{ixh}-1-ihx)e^{i\alpha x}h(x)dx\right|$. Note that for

$|x|<R$, we have $\dfrac{|e^{i\alpha h}-1-ihx|}{|h|}\to 0$, as $|h|\to 0$, because ix is the derivative of

$e^{i\alpha x}$ at $\alpha=0$. Moreover,

$$\left|\frac{1}{|h|}(e^{i\alpha h}-1-ihx)e^{i\alpha x}f(x)\right| \le ce^{R(|\zeta\alpha|+1)}|f(x)|$$

which is integrable, and so the D.C.T tells us that, in fact $\dfrac{|F(\alpha+h)-F(\alpha)-g(\alpha)|}{|h|}\to 0$.

Hence the proof is complete. ∎

Now, we give some propriety in Schwartz's space S using the main result.

3. THE SPACE S OF L.SCHWARTZ

We denote by S the Schwartz's space of all complex-valued functions $f(x)$ of real variable x , $-\infty < x < +\infty$, such that f is differentiable infinitely often, and for any integers p, q,

$$x^p f^{(q)}(x) \to 0 \quad as \quad |x| \to 0,$$

where $f^{(q)}$ denoting the q^{th} derivative of f . More details may be found in $[13-17]$.

Proposition 3.1 :

(1) If $f \in S(R)$, then $x^l f^{(m)}(x)$ is bounded, and belongs to $L_1(R)$, for any integers $l, m \ge 0$.

(2) If $f \in s$, then $\left(x^{(l)} f(x)\right)^{(m)}$ is bounded, and belongs to $L_1(R)$, for any integers $l, m \ge 0$.

Proof:

1) let $f \in S$, then for any integers $l, m \ge 0$, there is $M_0>0$ and $M_1>0$ such that

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$$\left|x^{l} f^{(m)}\right| < M_0 < +\infty. \quad (3.1)$$

and

$$\left|x^{l+2} f^{(m)}(x)\right| < M_1 < +\infty. \quad (3.2)$$

From (3.1) and (3.2), we have

$$\left|x^{l} f^{(m)}(x)\right| < \frac{M}{1+x^2} \in L_1(R).$$

2) This follows from (1), if we just use the rule for the differentiation of a product. Hence the proof is complete. ■

Theorem 3.2 : If $f(x) \in S(R)$, then the Fourier transform $\hat{f}(\alpha)$ belongs to $S(-\infty < x < +\infty)$.

Proof Let $f(x) \in S(R)$, then by (1) of proposition president we have $x^{l} f(x) \in L_1(R)$, for any integer $l \geq 0$, so that by theorem 2.1

$\hat{f}(\alpha)$ is differentiable infinitely often.

On the other hand, for l and m are positive integers, then by theorem 2.1, we have

$$\left(\hat{f}\right)^{(l)}(\alpha) = \int_{-\infty}^{+\infty} \left[(ix)^{(l)} f(x)\right] e^{i\alpha x} dx, \quad (3.3)$$

and by theorem 2.2, we have

$$\left|\alpha^{m} \left(\hat{f}\right)^{(l)}(\alpha)\right|$$

$$\leq \int_{-\infty}^{+\infty} \left\{(ix)^{(l)} f(x)\right\}^{(m)} dx. \quad (3.4)$$

Since $\left(x^{(l)} f(x)\right)^{(m)} \in L_1(R)$, from (3.4) and by Riemann Lebeague Theorem, we conclude that

$$\left|\left(\hat{f}\right)^{(l)}(\alpha)\right| = o\left(\frac{1}{|\alpha|^{m}}\right), \ as \ |\alpha| \to 0. \quad (3.5).$$

Hence the proof is complete. ■

Proposition 3.2 : Let $\Phi(x) \in S(R)$ and $f \in L_2(R)$ is such that $x f(x) \in L_2(R)$, and the Fourier transform is such that $\hat{f}(\xi) \in L_2(R)$, then $\exists g \in L_2(R)$ such that

$$\int f(x)\left(\frac{d}{dx}\Phi(x) + x\Phi(x)\right)dx = \int g(x)\Phi(x)dx$$

Proof : Since $x f(x) \in L_2(R)$ is given, we only need to find $\exists g_1 \in L_2(R)$ satisfying

$$\int_R f(x)\left(\frac{d}{dx}\Phi(x)\right)dx = \int_R g_1(x)\Phi(x)dx, \ \forall \Phi \in S(R)$$ and then

$g = g_1 + x f(x)$ will solve the problem. If we set $\Phi = \hat{\psi}$, as we can with $\psi \in S(R)$ uniquely determined, then $\frac{d\Phi}{dx} = i\,\xi \hat{\psi}(\xi)$, so

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$$\begin{aligned}\int_R f(x)\left(\frac{d}{dx}\Phi(x)\right)dx &= i\int_R \xi\hat{\psi}(x)dx\\ &= i\int_R f(x)\xi\psi(\xi)d\xi\\ &= \int_R g_1(\xi)\psi(\xi)d\xi\\ &= \int_R g_1(x)\Phi(x)dx\end{aligned}$$

where by definition $g_1 \in L_2(R)$ is the function with Fourier transform equal to $i\,\xi\hat{\psi}(\xi)$ in $L_2(R)$ by assumption. Hence the proof is complete. ■

Proposition 3.3: There exist an element $f \in S(R)$ with $\int_R |f|^2 dx = 1$, and

$$\int_R x^k f(x)dx = 0, \forall k.$$

Proof: If $f \in S(R)$, then it's Fourier transform is Schwartz and conversely. We also know that

$$\frac{d^k}{d\xi^k}\hat{f}(0) = (-i)^k \int_R x^k f(x)dx.$$

Thus we just have to arrange that $\hat{f}$ and all it's derivatives vanish at the origin. We do know that there is a non-trivial Schwartz function which vanishes outside the interval $[1,2]$ for instance. Taking this at the Fourier transform of cf and when choosing the positive constant so that $\|f\|_{L_2} = 1$, we get that

$$\int_R x^k f(x)dx = 0, \forall k.$$

Hence the proof is complete. ■

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